

A new type of diagrams for modules

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Abstract

We introduce a new type of diagrams and prove the existence of a particular one, the "central tuned diagram", with some optimal features, for finitely generated modules of certain categories. This is achieved by getting to the idea of "the virtual category" of a module. Important applications are specifically suggested to the modular representations of finite groups of Lie type.

1 Introduction

We prove the existence of a "virtual diagram" for every module in certain categories, for example that of A -modules of finite k -dimension over a Frobenius k -algebra A .

This is done by proving the existence of a particular, well defined diagram, to be called "the centrally tuned diagram", which is made so as to illustrate the longest (in radical length) indecomposable summands ("longest pillars") of radical sections (i.e., of sections of the type $Rad^i(M)/Rad^j(M)$) as the connected components of the relevant radical section of the diagram. The problem would not be so hard, if there were no isomorphic indecomposable summands in radical sections, in particular, isomorphic irreducibles in a radical layer: This is also why all previous efforts to obtain some module diagrams, that we know of, have essentially restricted themselves to this convenient special case.

To that purpose we introduce the notion of virtual radical (or, dually, socle) series of M (and, of course, of A itself), in which the simple constituents of the series are not just determined up to isomorphism but elementwise (set-theoretically). The easiest way to conceive the idea is through filtrations. Whenever we have a direct sum of isomorphic indecomposables, a realization is an (elementwise) choice of an expression of that sum; in the process of realization priority is given to the longest pillars. The realization of the minor pillars (in radical subsections of the radical sections of the major pillars) shall then be subjugated to the realizations of the major ones; this is a key idea, which we refer to as "vertical priority".

After those coordinated ("tuned") "realizations", we appeal to the relevant extension theory of S by N and its interpretation in terms of homomorphisms in $\text{Hom}_A(\Omega^1 S, N)$ (for A an artinian ring or a Frobenius algebra), monitored by means of the second virtual radical layer. In the same context we introduce the intuitively very important notion of proportion class of an extension. That theory is briefly reviewed with a fresh eye and some (may be) new insights in the next section.

All this can be much more neatly done in the frame of a new category we define in section 3, "the virtual category of a module". That sheds a completely new light and meaning in the radical (/socle) series, as well enables us to get to "virtual diagrams" in the general case.

Our thoughts with our module diagram extend to two directions: To highlight, (a) the section structure / the possible filtrations, (b) homomorphisms between modules, in particular automorphisms and endomorphisms. The case of automorphisms is of a crucial importance, as any homomorphism between modules may be viewed as the composition $\alpha \circ h \circ \beta$, where α, β automorphisms (α of a submodule-image, β of a factor module-coimage) and h a canonical homomorphism, the elucidation of which is the same as that of submodules and factor modules.

By doing that, it shall be possible to get to complicated diagrams from simpler subdiagrams, corresponding to their sections. In this respect, the assertion on the existence of such a diagram and its properties shall be of a crucial relevance.

We intend to investigate such possibilities and exemplify them properly in future work. Among others, we wish to investigate, the possibility of getting from relatively "nice" and simple diagrams for injective modules of a reductive, simply connected semisimple algebraic group G over an algebraically closed field of positive characteristic (in the defining characteristic) their restriction down to the subgroup $G(n)$ of G 's points over the field with p^n elements. This is discussed in the last section.

Notation 1 For a ring R we shall denote the i 'th power Jacobson radical by $J^i(R)$, while for the i 'th power Jacobson radical of a (left, when nothing else is said) R -module M we shall use both $J^i(R)M$ and $\text{rad}^i(M)$.

2 Ext, Hom traced on "virtual" radical (/socle) series

In this section, we approach the notion of "virtuality" in **radical** (resp. **socle**) **series** of a module set-theoretically, where everything shall be understood as

a submodule or subsection of the given module, together with some implicit identification.

All this is more properly organized in the next section, by introducing the concept of the "virtual category", as the idea that naturally enables us get a clear overview of the general module structure. In particular, we are thus getting overview of the modules Ext , as we are already going to see in this section for Ext^1 , in a 1-1 correspondence to specific set-theoretically specified (i.e., not just up to isomorphism) simple modules on the second radical layer, in the case of p.i.m.'s (or dually the second socle layer). That could previously somehow be done at most in case that there were no isomorphic simple modules on that layer.

After taking the consequence of this specified insight by getting to the idea of the virtual category, the next step is bound to be the idea of the virtual diagram, which is to be the subject of the next section.

Definition 2 *A **virtual radical** (resp. **socle**) **series** of a module is one where the simple constituents on each layer are not just meant as "isomorphic copies", but they have also been specified set-theoretically.*

Notice that the subsequent orders ("powers") of the radical, respectively of the socle, are submodules, thus set-theoretically well defined (with respect to the original module). However, when taking the quotient of radicals (resp. socles) of subsequent orders, in order to form the layers of the series, one defines them just up to isomorphism. We wish here to break with this tradition: **To this end, we consider the constituents of each layer set-theoretically as consisting of cosets of the next order radical (resp., of the previous order socle).** This subtle differentiation opens up for new insights and possibilities, as I am beginning to exhibit in this paper.

This set-theoretical specification at all layers necessitates some **identifications of sets**, which in this section are just introduced intuitively, but which shall finally lead us safely to the definition of the virtual category in the next section.

Condition 3 *By viewing the elements (of the irreducibles of) each radical layer as the biggest possible (: down to the last layer) cosets, it becomes possible to project them to the lesser cosets (by factoring out a submodule), enabling us to IDENTIFY the subsections of (f.ex. radical) radical sections to their canonically isomorphic sections, in particular identify any isomorphic pairs of the type $\text{rad}^i M / \text{rad}^j (M)$, $(\text{rad}^i M / \text{rad}^{j+\lambda} M) / (\text{rad}^j M / \text{rad}^{j+\lambda} M)$ ($i < j$) as well as their contents in a well-defined 1 – 1 correspondence.*

We are taking this identification for given throughout this work; it is, however, more proper to define a suitable category, where also these identifications are fully put into their right context. That is being done in the next section.

We note further that the "virtuality" of the contents of the radical layers is also conceivable through suitable (: compatible, i.e. **realizable**, see below) filtrations, a point that is important to keep in mind. **Indeed, virtual (radical/socle) series (and the corresponding virtual diagrams, which we are going to define) are, at least in case not all simple copies on the same layer are non-isomorphic, the only kind of picture of a module, that may properly enable us to follow and overview filtrations.**

It proves also expedient to our goals to introduce a new notion of a more confined inverse image, in cases of canonical epimorphisms from direct sums.

In our new, "virtual category" we are going to identify two (plus one) kinds of pairs of naturally isomorphic subsections of a given module M :

a. Pairs "of type q" $\{A/B, A/C \diagup B/C\}$ and

b. Pairs "of type s" $\left\{ \left(\bigoplus_{i=1}^n M_i \right) \diagup \left(\bigoplus_{i=1}^n S_i \right), \bigoplus_{i=1}^n M_i/S_i \right\}$, to which we shall

add all pairs "of type S" $\left\{ \tilde{\pi}^{-1}(N), \varpi^{-1}(N) \right\}$, i.e. preimages of canonical epimorphisms from direct sums and what we shall be calling "confined preimages", which will mean following :

Let $M = \bigoplus_{i=1}^n M_i$ be a certain decomposition of the section M of a given module K as a direct sum of indecomposables with each M_i possessing a certain submodule S_i ; consider the natural isomorphism $\sigma : \left(\bigoplus_{i=1}^n M_i \right) \diagup \left(\bigoplus_{i=1}^n S_i \right) \longrightarrow \bigoplus_{i=1}^n M_i/S_i$, and the canonical epimorphisms $\varpi : M \longrightarrow M \diagup \left(\bigoplus_{i=1}^n S_i \right)$ and $\pi_i : M_i \longrightarrow M_i/S_i$. Further, for any subset $J = \{j_1, \dots, j_s\}$ of $\{1, \dots, n\}$ define $\pi_J := (\pi_{j_1}, \dots, \pi_{j_s}) : \bigoplus_{j \in J} M_j \longrightarrow \bigoplus_{j \in J} M_j/S_j$ and let p_J be the usual direct sum projection corresponding to the index subset J , $\bigoplus_{i=1}^n M_i/S_i \longrightarrow \bigoplus_{j \in J} M_j/S_j$.

Assume, now, N to be any submodule of $M \diagup \left(\bigoplus_{i=1}^n S_i \right)$. Notice that, if we are "inside of a module K ", i.e. if M is a section of a module K , then $M \diagup \left(\bigoplus_{i=1}^n S_i \right)$ and $\bigoplus_{i=1}^n M_i/S_i$ are being "virtually" (i.e., in the virtual category \hat{W}_K of K) identified (as pairs "of type s").

Definition 4 Let J be the subset of $\{1, \dots, n\}$ which is minimal with the property that $\sigma(N)$ be contained in $\bigoplus_{j \in J} M_j/S_j$ (equivalently, that $\text{res}_N(\pi_J \circ \sigma)$ is injective). We define "the confined ϖ -preimage $\tilde{\varpi}^{-1}(N)$ of N " as, $\tilde{\varpi}^{-1}(N) := (\pi_J^{-1} \circ p_J \circ \sigma)(N)$ (being a submodule of $\bigoplus_{j \in J} M_j/S_j$).

In the following we shall just be using the notation $\tilde{\omega}^{-1}(N)$ for it, without further notification whenever the context is clear. Take care of the important fact, that this preimage depends on the choice of a decomposition $M = \bigoplus_{i=1}^n M_i$ into indecomposables (but not really on the ϖ -kernel $\bigoplus_{i=1}^n S_i$), or at least of the direct summand $\bigoplus_{j \in J} M_j$, unless all the indecomposables M_i , $i = 1, \dots, n$, are non-isomorphic, in which case the definition remains unambiguous, with no need for further specification.

The direct sums that are relevant in our context here are just ones that appear as sections of the module.

In order to get full advantage of virtual series, we must find an optimal and coordinated way of choosing the direct summands on each layer in cases of more than one isomorphic copies of irreducibles. This shall eventually lead us to the natural idea of a virtual diagram depicting the structure of the module.

We begin here with some preparatory revision and reinforcement (in virtual terms) of some very closely related facts.

In what follows we shall be considering a Frobenius k -algebra A (for example a finite-group algebra) and by "modules" we mean left A -modules (always finite dimensional as k -vector spaces). Of course, we are thinking of group algebras over fields of characteristic $p \neq 0$ dividing the order of the group, since otherwise, according to Maschke's theorem, the group algebra be semisimple - and hence uninteresting to us.

Let V be an A -module and consider a virtual radical (and a virtual socle) series of it.

Proposition 5 *If ϕ is a non-zero (A) -homomorphism $VJ(A) \rightarrow N$, then ϕ cannot factor through a projective A -module.*

Proof. *First proof:*

By isolating the relevant summand, we may reduce to the case that V is indecomposable.

Let, so, ϕ factor through P : $\phi = ba$, where $b : P \rightarrow N$.

We are first proving that then, ϕ does also factor through V 's injective hull, say Q . Consider V embedded as a submodule of Q ; then, apparently, $J(A)V \subseteq J(A)Q$.

$$\begin{array}{ccccc}
 Q & & \psi \rightarrow & P & \\
 \iota \uparrow & & a \nearrow & \downarrow b & \\
 VJ(A) & & \phi \twoheadrightarrow & N &
 \end{array}$$

Due to P 's injectivity we get $\psi : Q \rightarrow P$, such

\uparrow
 0

that $\psi\iota = a$, hence $(b\psi)\iota = \phi$.

Since N is simple, $b\psi : Q \longrightarrow N$ suggests $QJ(A) \subseteq \ker b\psi$, hence also $VJ(A) \subseteq \ker b\psi$, which necessarily means $\phi = 0$, a contradiction.

Second proof:

ϕ induces an isomorphism $N_1 \longrightarrow N$ for some simple submodule N_1 of $J(A)V/J^2(A)V$. In case ϕ factors through a projective P , say by $\iota : J(A)V \longrightarrow P$, that induced isomorphism $N_1 \xrightarrow{\sim} N$ necessarily factors through the one induced from $\text{Im}(\iota)$'s head. But since b induces an epimorphism from P 's head onto N , it becomes clear that $\text{Im}(\iota)$'s head lies in P 's head. Hence P 's head contains an isomorphic copy of N .

We are first proving that ϕ does also factor through N 's projective cover, say P_N .

$$\begin{array}{ccc}
 P & \xrightarrow{\quad} & P_N \\
 \iota \uparrow & b \searrow & \downarrow \\
 VJ(A) & \xrightarrow{\phi} & N
 \end{array}
 \quad \text{In case } P \text{ is not isomorphic to the projective cover}$$

\downarrow
 0

of N , then its head, as noticed containing an isomorphic copy of N , must also contain more irreducibles than that (see f.ex. [5, 6.25(i), p. 133]). On the other hand, by P 's projectivity we get a homomorphism $P \longrightarrow P_N$, which is surjective, since it induces an epimorphism onto P_N 's simple head N , and which must then split, due to P_N 's projectivity: $P = P_0 \oplus P'_N$, with $P'_N \cong P_N$ and $P_0 = \ker(P \longrightarrow P_N)$, another projective, as a direct summand of a projective. For the same reason the composite $\sigma : J(A)V \longrightarrow P_N$ is an epimorphism (as inducing an isomorphism $N_1 \xrightarrow{\sim} N$ onto the head of P_N), thus split, yielding $VJ(A) = \ker \sigma \oplus M$, with $M \cong P_N$, thus giving a composite monomorphism $\zeta : P_N \hookrightarrow J(A)V \hookrightarrow V$, which has to split, as P_N is injective too, since A has been assumed to be (symmetric, hence also) (quasi-)Frobenius. But this does also imply that the $Hd(\zeta(P_N)) = N_1$ (notice also here the "=" instead of " \cong ") is a direct summand in $Hd(V)$, contrary to the fact that N_1 is actually in the second layer of the virtual radical series of V . ■

Remark 6 *We wish here to emphasize that, by saying " $\phi : M \longrightarrow N$ factors through a projective" is meant that it may be written as a composition with some $\beta : P \longrightarrow N$, i.e. from P to N !*

We remind the reader that the Heller operators $\Omega^n V = \ker(\partial_{n-1} : P_{n-1} \longrightarrow P_{n-2})$ (for $n > 0$ and a minimal projective resolution of V - dually for $n < 0$) may only be defined in a category of finitely generated A -modules, where the Krull-Schmidt theorem holds and hence a unique minimal projective resolution is available; to this end, it is actually sufficient to assume that the ring A is (left-) artinian (see [5] for a proof), an assumption that we are keeping to here. Otherwise these operators are in general only definable in the stable category, i.e. up to a projective direct summand.

Lemma 7 *If ϕ is a non-zero $(A-)$ homomorphism $\Omega^n V \longrightarrow N$ (A a Frobenius k -algebra, N simple, $n > 0$), then ϕ cannot factor through a projective A -module.*

Proof. By taking a minimal projective resolution of V , $\dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow$

$V \longrightarrow 0$, $\Omega^n V$ is isomorphic to $\ker(\partial_{n-1} : P_{n-1} \longrightarrow P_{n-2})$, embedded in P_{n-1} through, say, ι . Appealing to the Frobenius property, P is injective too, which forces a to factor through $\psi : P_{n-1} \longrightarrow P$. We may then again proceed in two ways:

First proof:

$$\begin{array}{ccccc} P_{n-1} & \xrightarrow{\psi} & P & & \\ \iota \uparrow & a \nearrow & \downarrow b & & \\ \Omega^n V & \xrightarrow{\phi} & N & \text{Due to the minimality of the resolution, } \Omega^n V \subseteq & \\ \uparrow & & & & \\ 0 & & & & \end{array}$$

$J(A)P_{n-1}$; on the other hand, since $N = \text{Im}(b\psi)$ is simple, we get $J(A)P_{n-1} \subseteq \ker(b\psi)$, therefore also is $b\psi\iota$ (meaning here, the restriction of $b\psi$ to $\Omega^n V$) equal to 0, hence $\phi = 0$, contrary to our assumption.

Second proof:

By arguing in the same way as in the second proof of the proposition above, we get by assuming that ϕ factors through a projective P , that $\Omega^n V$ has a projective direct summand isomorphic to the projective cover of N , which contradicts the minimality of our resolution. ■

By dualizing (either by starting with a minimal injective resolution and mirror the above proof or by using the isomorphism $\Omega^{-n} V \cong \Omega^n (V^*)^*$ and the lemma), we get the following

Corollary 8 *If ϕ is a non-zero $(A-)$ homomorphism $N \longrightarrow \Omega^{-n} V$ (A left artinian, N simple, $n > 0$), then ϕ cannot factor through a projective A -module.*

Let now A be a Frobenius algebra, in which case the category ${}_A \text{mod}$ of (left) A -modules is a Frobenius category.

Furthermore we are now for a moment shifting our contemplation from the category ${}_A \text{mod}$ into the (triangulated) stable category ${}_A st \text{mod}$, which still has the same objects as ${}_A \text{mod}$ (although we are now getting much bigger/fewer isomorphism classes of them), but whose morphisms $\underline{Hom}_A(,)$ are obtained as equivalence classes in $Hom_A(,)$ modulo the "ideal" (in the sense of an additive category) $PHom(,)$, consisting of those homomorphisms, that factor through a projective module; i.e., given two A -modules M, N , $\underline{Hom}_A(M, N) = Hom_A(M, N)/PHom_A(M, N)$. This is a triangulated category with translation functor $T = \Omega^{-1}$. A very important point in considering the stable category is the fact that

$$Ext_A^n(M, N) \cong \underline{Hom}_A(\Omega^n M, N) \cong \underline{Hom}_A(M, \Omega^{-n} N) \quad (1)$$

for any $n \in \mathbb{Z}$ (See f.ex. [4, 5.1, also 5.2 & 4.4(v)]).

Up to this point it would be enough to have a Frobenius category, such as the category ${}_A \text{mod}$ of modules over a Frobenius algebra A . If we now specialize in the case, where A is a group algebra kG over a finite group G , which also is our main motivation, then we have an algebra that is not only artinian and Frobenius, but also symmetric. For a symmetric k -algebra A the p.i.m.'s (principal indecomposable modules, i.e. the projective covers [being here the same as the injective hulls] of its simple modules) have isomorphic head and socle (see f.ex. [13, I 7.5(iii)]); for a more general category-theoretic set-up and approach to the stable category and related topics see f.ex. [11], for more specific details as to the triangulization of the stable category of modules over a finite-group algebra see [4].

Our Lemma with its Corollary above, together with relation (1), imply the following

Proposition 9 *Given a Frobenius algebra A , the A -modules M, N , where N is simple, for $n > 0$ we have $\text{Ext}_A^n(M, N) \cong \text{Hom}_A(\Omega^n M, N)$ and $\text{Ext}_A^n(N, M) \cong \text{Hom}_A(N, \Omega^{-n} M)$.*

Proof. Our Lemma, resp. its Corollary, gives $\text{PHom}_A(\Omega^n M, N) = 0$ and, respectively, $\text{PHom}_A(N, \Omega^{-n} M) = 0$ - which, by virtue of the two relations (1), yield the results. ■

Remark 10 *We note that, by following upon the line of the proof of the above Lemma 5, we can show the proposition independently of the relations (1), for any artinian algebra; see for example [1, 2.5.4].*

We are going to pursue this correspondence further, by following it along the virtual series in detail: *a possibility which is only there, if we are considering the virtual instead of the usual series.*

Definition 11 *We call two extensions $0 \rightarrow N_\kappa \rightarrow B_\kappa \rightarrow S \rightarrow 0$, $\kappa = 1, 2$, "proportional" if there are isomorphisms $\sigma : N_1 \rightarrow N_2$ and $B_1 \rightarrow B_2$, such*

$$\begin{array}{ccccccc} 0 & \rightarrow & N_1 & \rightarrow & B_1 & \rightarrow & S \rightarrow 0 \\ & & \downarrow \sigma & & \downarrow & & \parallel \\ 0 & \rightarrow & N_2 & \rightarrow & B_2 & \rightarrow & S \rightarrow 0 \end{array}$$

that the following diagram commute:

We shall call σ their "proportion"; in case N is simple, σ corresponds to a non-zero element of the field k , as then $\text{Aut}_A(N) \cong k^$ (Schur's Lemma).*

Notice that in this definition we did not assume N_κ ($\kappa = 1, 2$) and S to be simple; however we are primarily going to apply the concept in that case. This defines an equivalence relation on the set of extensions of S by N , the classes of which we shall call "proportionality classes".

Notation 12 Let now V be an A -module, with simple head S (forcing V to be indecomposable) and let its second layer of virtual radical series be $VJ(A)/VJ^2(A) = \bigoplus_{i=1}^r N_i \oplus W$ ($J(A)$ denotes the Jacobson radical of A), where $N_1 \cong N_2 \cong \dots \cong N_r \cong N$ simple A -modules and W is a sum of irreducibles, all non-isomorphic to N . Notice that we have put " $=$ " **and not** " \cong ": that is due to our set-theoretically virtual approach. For $\kappa = 1, \dots, r$ we get a factor module $B_\kappa = \frac{S}{N_\kappa}$, which we may specify more precisely as follows: Let $VJ(A)/VJ^2(A) = N_\kappa \oplus R_\kappa$ and let $\pi : VJ(A) \rightarrow VJ(A)/VJ^2(A)$ be the natural epimorphism. Then $B_\kappa := V/\pi^{-1}(R_\kappa)$.

As it is well-known, the elements of $Ext_A^n(M, N)$ are in 1-1 correspondence with the set of equivalence classes of n -extensions of M by N (see [4, Th. 6.3, p. 29]). Due to the isomorphism above, we do also get an 1-1 correspondence between $\underline{Hom}_A(\Omega^n M, N)$ and the set of equivalence classes of n -extensions of M by N ; that correspondence is in terms of $\underline{Hom}_A(\Omega^n M, N)$ precisely specified in [1, 2.6, p. 40]. We shall mainly be concerned with this correspondence for $n = 1$ and for modules over a symmetric algebra, in which case we also have Proposition 7.

Adjusting to the standard notation, call ζ_κ the element of $Ext_A^1(M, N)$, which corresponds to the canonical surjection $\widehat{\zeta}_\kappa : \Omega^1(S) \rightarrow N_\kappa$, and let L_κ designate its kernel.

Let us now consider the standard minimal projective resolution of S , where we will denote the projective cover of S as P_0 , with the whole resolution taken in a set-theoretically virtual way, thus enabling us f.ex. to consider $P_0/P_0J(A)$ (now, of course, $\Omega^1 S$ is the same as P_0 's radical $P_0J(A)$) as **equal** and **not just isomorphic** to S - and so on, for the whole complex! It may appear wierd doing so, while moving "up the hill" in subsequent epimorphisms (which we want to be able to identify as "the natural ones", i.e. as gotten by taking the quotients by submodules, which allows for virtuality), but we may always start up at the necessary level and move "downhill", down to modules consisting of (ever bigger) cosets.

Let us use the notation above for the module P_0 in place of V , where we now have $L_\kappa = \pi^{-1}(R_\kappa)$, in the above notation.

$$\begin{array}{ccccccccccccccc}
\rightarrow & P_1 & \rightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\quad} & \underline{P_1} & \xrightarrow{\quad} & P_0 & \rightarrow & S & \rightarrow & 0 \\
& & & \downarrow & & \downarrow & & q \searrow & & \nearrow & & \downarrow & & \\
& & & & & \mu_1 \downarrow & & \Omega^1 S & & & & \mu_0 \downarrow & & \parallel \\
& & & \downarrow & & \downarrow & & \swarrow \widehat{\zeta}_\kappa & & & & \downarrow & & \\
& & & 0 & \rightarrow & N_\kappa & \xrightarrow{\quad} & \dots & \rightarrow & B_\kappa & \rightarrow & S & \rightarrow & 0
\end{array}$$

We know that any two different homomorphisms $\widehat{\zeta}_\kappa : \Omega^1 S \longrightarrow N$, $\kappa=1,2$, represent different classes in $Ext_A^1(S, N)$, hence also non-equivalent extensions; of course, those different homomorphisms are in 1-1 correspondence to the homomorphisms $\overline{\zeta}_\kappa : Hd(\Omega^1 S) \longrightarrow N$, which are in turn up to an automorphism of N (meaning here up to an automorphism of N_κ) determined by $\ker(\overline{\zeta}_\kappa)$. But "up to an automorphism of N_κ " means "up to a proportion", in view of our definition; i.e. each direct summand in $Hd(\Omega^1 S)$ determines a proportionality class. Hence we get the following

Proposition 13 *With the above notation, two homomorphisms in $Hom_A(\Omega^1 S, N)$ correspond to proportional extensions of S by N (equivalently, they correspond to each other's multiple in $Ext_A^1(S, N)$) iff their induced homomorphisms $Hd(\Omega^1 S) \longrightarrow N$ have the same kernel. We may thus have a precisely determined A -homomorphic correspondence $Ext_A^1(S, N) \xleftrightarrow{\sim} Hom_A(Hd(\Omega^1 S), N) \cong k^r$ (by Schur's lemma), where r is the number of direct summands of $Hd(\Omega^1 S)$, that are isomorphic to N , whereby the proportionality class of an extension B of P_0 's head S by N is determined by the direct summand isomorphic to N in $Hd(\Omega^1 S)$ that is involved ($:$ is not in the kernel).*

Proposition 14 *For a module V as above, i.e. with simple head S and with $J(A)V/J^2(A)V = \bigoplus_{i=1}^r N_i \oplus W$, where $N_1 \cong N_2 \cong \dots \cong N_r \cong N$ simple A -modules and W is a sum of irreducibles, all non-isomorphic to N , the elements in $Ext_A^1(S, N)$ corresponding to the equivalence classes of the r extensions $0 \longrightarrow N_\kappa \longrightarrow B_\kappa \longrightarrow S \longrightarrow 0$ ($N_\kappa \cong N$) are again k -linearly independent.*

Theorem 15 *Conversely, if a module V with the simple S in its head has r factor modules of the type $\begin{smallmatrix} S \\ N_\kappa \end{smallmatrix}$, with $N_\kappa \cong N$, $\kappa = 1, \dots, r$, with that S on the head, so that these r extensions correspond to r k -linearly independent elements in $Ext_A^1(S, N)$, then we have an injection $\bigoplus_{i=1}^r N_\kappa \hookrightarrow J(A)V/J^2(A)V$, i.e., we get to a non-split extension $0 \longrightarrow \bigoplus_{i=1}^r N_\kappa \longrightarrow B \longrightarrow S \longrightarrow 0$, where B is a factor module of V .*

Proof. By pointing out that we have an epimorphism onto V from the pro-

jective cover P_S of S , we may substitute P_S for V and study the lifts of these extensions there; the reason why this epimorphism exists is that the projective

cover of V is just P_S , i.e. the same as the projective cover of its head S (see [5, 6.23 or 6.25 (ii)]). Speaking in terms of virtual radical layers, we may lift the chosen N_κ 's along $P_S \twoheadrightarrow V$, set theoretically and elementwise, to ones now consisting of bigger cosets as elements, **in a procedure that clearly respects the extension classes of the B_κ 's.**

The converse is clear. ■

This proposition may of course be dualized, to get a similar one in terms of the virtual socle series:

3 Module Diagrams

As we intuitively did in the previous section, so also in our new, "virtual category" of a given module K we are going to identify two (plus one) kinds of pairs of naturally isomorphic subsections of the given module:

- a. Pairs "of type q" $\{A/B, A/C/B/C\}$ and
- b. Pairs "of type s" $\left\{ \left(\bigoplus_{i=1}^n M_i \right) / \left(\bigoplus_{i=1}^n S_i \right), \bigoplus_{i=1}^n M_i / S_i \right\}$, to which we shall add all pairs "of type S" $\left\{ \tilde{\pi}^{-1}(N), \varpi^{-1}(N) \right\}$, i.e. preimages of canonical epimorphisms from direct sums and their "confined preimages".

The direct sums that are going to appear inside our virtual category are only sums that appear as sections of the module; otherwise we can apparently not define sums in general in such a category.

Definition 16 *Given a module K , we define its "virtual category" \hat{W}_K , whose*

objects are obtained from the family of all of K 's sections, sections of sections and so on, after identifying (also elementwise meant) all pairs "of type q"

$\{A/B, A/C/B/C\}$ and also all pairs "of type s" $\left\{ \left(\bigoplus_{i=1}^n M_i \right) / \left(\bigoplus_{i=1}^n S_i \right), \bigoplus_{i=1}^n M_i / S_i \right\}$ of naturally isomorphic subsections of K , as well as pairs "of type S" $\left\{ \tilde{\varpi}^{-1}(N), \pi^{-1}(N) \right\}$ (see previous definition), and in which morphisms are the ones induced by the module homomorphisms between the sections.

It is immediate to see that \hat{W}_K is an exact category.

We shall usually denote the objects of the virtual category \hat{W}_K (to be called "virtual sections" of K) with the same letters as the sections themselves, although we shall be considering them as objects of the virtual category of K , indeed corresponding to equivalence classes of subsections, resulting from the

identifications that we applied on the sections of K . This convention shall always be implied in the following, with no further notification.

We may also notice that in the case of identifications of type S it is more handy to use the confined preimages as representatives of the virtual sections in question.

In fact the previous chapter, where we have defined "virtuality" in a set-theoretic way (which may easily be shown to be equivalent to that of the virtual category), may also be viewed in the framework of the virtual category.

Let now A be a Frobenius k -algebra from now on.

As we have seen (Theorem 12), whenever we have an indecomposable with simple head, that simple head forms non-split extensions with all virtual irreducibles on the next radical layer, extensions which are also k -linearly independent in case they are isomorphic; further, if the indecomposable is projective (i.e., a p.i.m.), those extensions correspond to a k -basis of the corresponding module of type $Ext_A^1(,)$.

We shall in the following be working in a suitable category of modules, which may either be that of the finitely generated modules over such a Frobenius k -algebra A , or of the finitely generated rational G -modules, where G is a reductive algebraic group, or even a truncated subcategory $C(\pi)$ of that, for some saturated subset π of $X(T)_+$ [10, chapter A]. Whenever we come to representations of algebraic groups, if not specified, our notation shall be that of [10].

So henceforth, whenever we just refer to "a module", if not specified, we shall mean a module in any of these categories. **When working with (sub)sections of a particular module, that shall automatically be done in the context of its virtual category, in which subsequently all equalities have to be understood.**

It is very important for us, that also in the case of representations of reductive algebraic groups we do have some propositions quite similar to the last two propositions (of section 2), in which we may now use the Weyl modules and their duals instead of the p.i.m.'s that were used for modules over a Frobenius algebra (see f.ex. [10, II, 2.12(4) combined with 2.14 prop.&(4)]) - leading, by a similar to the above procedure, to:

Proposition 17 *For $\lambda, \mu \in X(T)_+$, with $\mu \not\prec \lambda$, we have a precisely determined G -homomorphic correspondence $Ext_G^1(L(\lambda), L(\mu)) \xleftrightarrow{\sim} Hom_G(radV(\lambda), L(\mu)) \xleftrightarrow{\sim} Hom_G(radV(\lambda)/rad^2V(\lambda), L(\mu)) \xleftrightarrow{\sim} k^r$ for some $r \geq 0$, as well as $Ext_G^1(L(\mu), L(\lambda)) \xleftrightarrow{\sim} Hom_G(L(\mu), H^0(\lambda)/socH^0(\lambda)) \xleftrightarrow{\sim} Hom_G(L(\mu), soc^2H^0(\lambda)/socH^0(\lambda)) \xleftrightarrow{\sim} k^r$, while also $Ext_G^1(L(\lambda), L(\mu)) \equiv Ext_G^1(L(\mu), L(\lambda))$.*

To start moving toward a diagram of an indecomposable module, we may attach a vertex to each simple at all layers, and start by joining the vertex corresponding to the simple head with all the vertices of the second layer, give

the edges an "upward" orientation, and attach to each of them the corresponding extension class (equivalently, the corresponding elements of Ext_A^1).

We may proceed this way, by taking the indecomposable subfactors with the simple heads of the subsequent layers and join each simple head to the essential irreducibles of the next radical layer of the relevant indecomposable sector, but such a process raises many subtle conditions, to make it possible.

We have a freedom of choice on "essential" irreducible summands, whenever there are multiple isomorphic copies on a layer, resulting in a variation of the extension classes, according to Theorem . However it is very important to make a properly coordinated choice of virtual irreducibles on the different layers, so as to make the process possible, and also get to a nicely "tuned" diagram, instead of a possibly chaotic picture. But what does it exactly mean for a diagram to be thus "nice"? We are thus lead to the notion of a "tuned" diagram:

Definition 18 We call a (virtual) diagram of a module M , that is associated to a virtual radical series, a **tuned** one, if the following condition is satisfied: For $0 \leq i < j$, any indecomposable summand of the "radical section" $\text{rad}^i M / \text{rad}^j M$ either precisely corresponds to some connected component of the corresponding radical section of the diagram (: "is visible on the diagram") or is a submodule of a direct sum of visible isomorphic direct summands of that section, i.e. ones that "virtually correspond" to (: "are realized by") some connected components of the corresponding graph section. Such indecomposable summands will be called " $\{i, j\}$ -pillars", of **height** at most $j - i$. An $\{i, j\}$ -**colonnade** is a maximal direct sum of (at least two) isomorphic $\{i, j\}$ -pillars. A pillar with no other isomorphic pillars in the same radical section is called a **single pillar**. A **specification of a colonnade** is any choice of specific pillars (of which we are thus also getting an "**illustration**", as they become **visible**) for the expression of the colonnade as a direct sum. A pillar B is "dominated" (\leq) by another one A , if in any diagrammatic illustration of the module M , in which the pillar A becomes visible, the pillar B is depicted by a subdiagram of the diagram section depicting A . In this context we shall be talking of "maximally/minimally dominating pillars in a (radical) section". Alternatively (conversely) we shall also describe that condition by telling that A **overcoats** B .

A single pillar of a radical section is an indecomposable direct summand with no other isomorphic summand in the same radical section. A **peak** of a colonnade is a single pillar that overcoats a pillar of the colonnade. As suggested above, we shall call a pillar or, more generally, a subsection of M , a **visible** one (on the diagram), if it properly (: virtually) corresponds to a certain subdiagram.

In the case of just a virtual radical or socle series of M , we shall say that a (virtual) section is visible or realizable on it, if all its simple (virtual) sections are included in it.

So, we might speak of visible sections, hoping also to somehow get able to gain an overview of the non-visible ones.

Conversely, we might ask: "When does a "locus" of the virtual radical/socle series (respectively, of a virtual diagram) correspond to a section of the module?" Our term in that case shall be that the subdiagram (or locus from the virtual radical/socle series) may or may not be **realizable**, where the virtual section that realizes the locus "overcoats" it.

A fundamental merit of a "good" virtual diagram shall be the optimal realizability of such loci and, conversely, the visibility of any subsection of the module or of a "parallel" isomorphic copy of it, meaning that they belong to the same colonnade.

Definition 19 We introduce a "weak" partial ordering " \sqsubseteq " in the set Π of colonnades of a module M , meaning that there is a specification of the "bigger", where at least one pillar dominates a specification of the "smaller" one. We shall be especially interested in the **maximal colonnades** with respect to this ordering.

After introducing all these key-notions, we shall prove some lemmas, that shall lay the way for the proof of the existence theorem for virtual diagrams.

Lemma 20 No two maximally dominating pillars of a module K may have a (virtually!) common section.

Proof. The question is easily seen to boil down to the following:

"Let K be a module of radical length $\kappa + 1$, and M, N a pair of pillars of K , where M is a direct summand of $K/\text{rad}^{i+s}K$, N of $\text{rad}^i K$, both of maximal radical length (resp., $i + s$ and $\kappa - i + 1$), for some $i > 1$, $s > 0$ with $i + s < \kappa + 1$, and such that M, N have a common section S , meaning here that S is virtually a submodule of M and a factor module of N , and S in the Loewy series "extends over" all the s layers of K 's radical section $\text{rad}^i K/\text{rad}^{i+s}K$; we may furthermore easily reduce to the case where $S = \text{rad}^i M = N/\text{rad}^s N \subseteq \text{rad}^i K/\text{rad}^{i+s}K$ (always virtually meant). Then neither M or N can be a maximally dominating pillar of M ."

The condition for M and N does not only suggest the existence of a homomorphism $\varphi : N \rightarrow M$, where $\ker \varphi = \text{Im } \varphi = S$, but here we do also have something analogue to the inclusion, considered as a special kind of monomorphism. This analogue becomes only possible in the frame of our virtual category, in which we have (the possibility) to see everything also set-theoretically at the same time, and not just up to isomorphism.

By the definition of S it is clear that it is realizable in T , inasmuch as M and N are so.

In my primitive drawing above I am trying to visualize the situation by denoting the location of S , shared in common by both M and N , with $\#\#$, that belonging only to M by vertical lines $|||$, the one exclusively belonging to N by horizontal lines $==$.

Let us first notice that, by going from the radical to the socle series, the location of section S shall still extend over "the same" s layers, because: (a) The socle series length of S and the indecomposables M and N remain the same as that of radical series, and (b) nothing can "fall down", so as to extend the length $i + s$ of M in K , because that would contradict the indecomposability of M , unless M as a whole be shifted downward, taking also S down with it; but that would necessarily decrease the length $\kappa - i + 1$ of the indecomposable N , a contradiction.

Set $K/rad^{i+s}K = M \oplus M_0$, implying $rad^i K/rad^{i+s}K = rad^i M \oplus rad^i M_0 = S \oplus S_0$, where $S_0 := rad^i M_0$. We have, further, the following virtual equalities: $K/rad^i K = K/rad^{i+s}K / rad^i K/rad^{i+s}K = (M \oplus M_0) / (S \oplus S_0) = M/S \oplus M_0/S_0 = M' \oplus M'_0$, where we have put $M' := M/S$ and $M'_0 := M_0/S_0$. Similarly, by writing $rad^i K = N \oplus N_0$, $rad^s N = N'$, $rad^s N_0 = N'_0$ we get $rad^{i+s}K = N' \oplus N'_0$.

Consider now the canonical epimorphisms $\pi : K \rightarrow K/\text{rad}^i K$, $\pi_1 : K \rightarrow K/\text{rad}^{i+s} K$, $\pi_2 : K/\text{rad}^{i+s} K \rightarrow K/\text{rad}^i K$; we are now going to utilize the notion of confined preimage.

Clearly $\pi = \pi_2 \circ \pi_1$, therefore $\tilde{\pi}^{-1}(M') = \tilde{\pi}_1^{-1}(\tilde{\pi}_2^{-1}(M')) = \tilde{\pi}_1^{-1}(M)$, the intersection of which with $rad^{i+s}K$ has to be the same as that of $\tilde{\pi}_0^{-1}(S)$, where $\pi_0 : rad^i K \rightarrow rad^i K / rad^{i+s} K$, inasmuch as $s > 0$, i.e. $\tilde{\pi}_1^{-1}(M) \cap rad^{i+s} K = \tilde{\pi}_0^{-1}(S) \cap rad^{i+s} K = N'$. However this means that $\tilde{\pi}_1^{-1}(M)$ is a module Λ , that precisely overcoats M and N of Loewy length $\kappa + 1$ and is clearly indecomposable; as soon as we prove this also to be a direct summand of K , then it shall be a pillar, that properly dominates over both M and N , contrary to their maximality.

Consider the (virtual) exact sequence $0 \rightarrow N' \oplus N'_0 \rightarrow K \rightarrow M \oplus M_0 \rightarrow 0$, where the first map is an inclusion and the second is $\pi_1 : K \rightarrow K/\text{rad}^{i+s}K (= M \oplus M_0)$; therefore by looking at the relevant confined preimage we get $\tilde{\pi}_1^{-1}(M) \cap \text{rad}^{i+s}K = N'$, where $\text{rad}^{i+s}K = N' \oplus N'_0$, implies $\tilde{\pi}_1^{-1}(M_0) \cap \text{rad}^{i+s}K = N'_0$, meaning that Λ is a direct summand of K , as claimed - leading to the suggested contradiction.

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It is easily seen that the module Λ obtained in the proof of the previous lemma is indeed the cup product (seen as Yoneda splice, see f.ex. [4, 6 (8), p. 35]) of the extensions $0 \rightarrow N' \rightarrow N \rightarrow M \rightarrow M' \rightarrow 0$ (where $N \rightarrow M$ is the map φ above) and $0 \rightarrow N' \rightarrow N \rightarrow S \rightarrow 0$ $0 \rightarrow S \rightarrow M \rightarrow M' \rightarrow 0$.

The map $\varphi : N \rightarrow M$ above in an intuitively very meaningful way generalizes two special cases of homomorphisms: namely, inclusion and canonical epimorphism. Such a generalization is only conceivable in the frame of a virtual category; we shall call them "canonical homomorphisms":

Definition 21 *Given two virtual sections M and N of a module K and a third one, say S , that is embedded in M and on which we, on the other hand, have a canonical epimorphism $N \twoheadrightarrow S$, we call their composition $N \twoheadrightarrow S \hookrightarrow M$ a **canonical homomorphism (over S)**. We may as well define the "**strictly virtual category**", in which only canonical homomorphisms are allowed as morphisms.*

It is obvious from the definition that inclusions and canonical epimorphisms are just special cases of canonical homomorphisms.

Lemma 22 *If L_1 and L_2 are visible summands of some radical (/socle) series sections, of Loewy length l_1 and l_2 respectively, with only one (visible) common section at their top, i.e. a common factor T (not necessarily simple!), then the pullback of the canonical epimorphisms $L_1 \rightarrow T$, $L_2 \rightarrow T$ is visible too.*

More generally, if we have canonical epimorphisms $L_i \rightarrow T$, $i = 1, \dots, s$, with all L_i and T visible, then they determine uniquely a section, that precisely overcoats them.

Proof. Assume first $l_1 = l_2$; then their sum $L_1 + L_2$, making sense inside the least radical series section that overcoats them, is actually the pullback. If $l_1 < l_2$, amputate L_2 to get to the first case and then use the technique of the previous proof to get the section that precisely overcoats L_1 and L_2 . ■

Observe that in case $l_1 \neq l_2$, we cannot make any sense of the sum $L_1 + L_2$ in the given virtual category, therefore we amputate the longest (say, L_2) down to, say, L_0 , make use of the first case of the proof of this Lemma to get a "virtual pullback" L and, subsequently, get the ("canonical", while all maps involved are so) cup product over L_0 of L and L_2 . That is actually the pullback of the canonical epimorphisms $L_1 \rightarrow T$, $L_2 \rightarrow T$.

By dualizing the arguments, we get the dual of the above:

Lemma 23 *If L_1 and L_2 are visible summands of some radical (/socle) series sections, of Loewy length l_1 and l_2 respectively, with only one (visible) common section at their bottom, i.e. a common virtual submodule T , then the pushout of the embeddings $T \rightarrow L_1$, $T \rightarrow L_2$ is visible too.*

More generally, if we have embeddings $T \rightarrow L_i$, $i = 1, \dots, s$, with all L_i 's and T visible, then they determine uniquely a section, that precisely overcoats them.

Proposition 24 *We may to every finitely generated A -module M attach a tuned radical diagram.*

Proof. *SKETCH OF PROOF:*

Clearly, it suffices to show it for an indecomposable module M .

We proceed to construct such a diagram:

Take all peaks and all maximal colonnades. Prove that they cannot overlap. Investigate possible differentiation of some realizations of maximal colonnades, prefer the ones that highlight them. "VERTICAL PRIORITY".

Consider then for any " \sqsubseteq "-maximal $\{i, j\}$ -colonnade the " \leq "-maximal (maximally dominating) indecomposable $\{i - 1, j + 1\}$ -overcoat (where, if either $i - 1$ or $j + 1$ layer non-existing, take i , resp. j instead) precisely overcoating the colonnade on the respective radical (socle) series section, call them "**pillar bricks**"; it follows easily that they can only overlap on their endlayers, same or opposite direction (use maximality of colonnades).

Our construction ensures that we do not mess with pillar-realizations, ensuring "vertical priority", in the sense that the "broader" (radical) section consideration comes before the lesser, narrower ones in the realization process.

Inside each brick, we consider the maximal indecomposables with one of the (virtually determined) irreducibles of the head as their simple head. We repeat the process in narrower radical sections inside the radical section of each maximal pillar, as many times as possible: That means, repeating the process inside the pillars of the maximal colonnades.

By taking all the indecomposables with simple head, we join the simple head by an edge with all irreducibles on its next radical layer, while attaching the proper extension-class to it.

Thus we get a diagram, which can easily be shown to be an "optimally tuned" one, in the sense of "vertical priority". ■

Conclusion 25 *We achieve a final **diagram** out of a "radical" one in the following way:*

Inside the virtual category of a given module M , we may then identify the virtual irreducibles of the radical series as contents of the socle series too, not necessarily at the same layers: The virtual irreducibles that are "hanging loosely" from the preceding radical layer shall fall down to a lower layer of the socle series. The key background property for this situation is the existence in a radical section of height s of a direct summand of Loewy length smaller than s . These provide the missing edges of the module diagram, by considering (dually now) the indecomposable sections with a simple socle in the subsequent socle layer; those indecomposables are the same as the ones we considered through the virtual radical series, inside the categories of modules we are considering. Thus we get to the important

Theorem 26 *We may to every finitely generated A -module M attach a certain type of a tuned diagram, to be called **a central diagram**. In particular, there is always a tuned diagram.*

Thinking of the virtual radical, resp. socle, series in terms of compatible filtrations, we may read such filtrations either "downward", i.e. from the bigger to the smaller (as in radical series), or upward (like the socle series). In the former case, a step is non-split whenever the next term loses a (virtual) irreducible of the next radical layer; dually in the latter case, a step upward is non-split whenever the next term gains a (virtual) irreducible of the next (:higher) socle layer.

It is not difficult to prove that it is possible to make a diagram that somehow highlights (: makes it visible) any particular section of the module. A lot of relevant questions arise: Can we consider the totality of tuned diagrams for a module? May some of them "lead" to all others - or to a family of them?

The Theorem of existence of a tuned diagram can become a powerful tool in many situations. In my following research I shall try to illustrate this by some important examples.

It may for example become a powerful tool in many cases where we restrict modules to subcategories/subgroups, also depending on the other information we may have, in particular about the restriction of the Ext-functors and other relevant specific knowledge.

We close this section by stressing that the present article has not yet reached its final edition.

4 An important field of application.

Here we are now going to outline our strategy for the determination of virtual diagrams for the p.i.m.'s of the family of groups $SL(3, p)$, with $p > 3$. That strategy may be pursued in the more general case of the above mentioned truncated category $C(\pi)$, closely related to the representation theory of finite groups of Lie type.

Particularly important in our considerations is the following result on truncated categories ([10, A.10, p. 393]):

Proposition 27 *Let $\pi \subset X(T)_+$ be saturated, V, V' be G -modules in $C(\pi)$. Then for all $i \geq 0$ there are isomorphisms $Ext_{C(\pi)}^i(V, V') \cong Ext_G^i(V, V')$.*

Even more important for our goals is the following Proposition, which was already proved in [6, Theorem 7.4], but of which Henning Haahr Andersen in [8, Proposition 2.7] gave a new elegant and self-contained proof:

Proposition 28 *Let G be an almost simple algebraic group over an algebraically closed field k of positive characteristic p , and assume that G is defined and split over the prime field F_p ; we denote as $G(n)$ the finite group consisting of the*

points of G over the field with p^n elements. Then for $\lambda, \mu \in X_n(T)$, the restriction map $\text{Ext}_G^1(L(\mu), L(\lambda)) \longrightarrow \text{Ext}_{G(n)}^1(L(\mu), L(\lambda))$ is injective.

However, by going through that proof we can confirm that, with a slight adaptation, we deduce the following stronger version, about which it has though to be remarked that it only makes full sense in the frame of the virtual category of the G -module E (and its restriction down to $G(n)$):

Proposition 29 *Same notation as above, except that we now allow for any $\lambda, \mu \in X(T)$, instead. Let $0 \rightarrow L(\lambda) \rightarrow E \rightarrow L(\mu) \rightarrow 0$ be a non-split G -extension. Then is $\text{Hom}_{G(n)}(L(\mu), E) = 0$, which implies that no simple composition factors of E as a $G(n)$ -module, that stem from the restriction of $L(\mu)$, can be in the socle. This has to be understood in the context of the virtual socle series of the G -module E and its restriction down to $G(n)$.*

This last proposition is going to be very important in our pursuit of diagrams for modular representations of finite groups of Lie type, as it is giving us a kind of thread coming by restriction from the diagrams for representations (typically the Weyl modules or their duals) algebraic groups down to the groups $G(n)$. We are then in many cases able to get some "good filtrations" of tensor products of suitable simple modules with the Steinberg module $St_n = H^0((p^n - 1)\rho)$, of which the corresponding G_n -injective hull $Q_n(\lambda)$ is proven to be a direct summand, by means of the well known G_n - (and $G(n)$ -) injectivity of St_n , and also of the fact that $Q_n(\lambda)$ is known to have a G -module structure for all $\lambda \in X_n(T)$ for all $p \geq 2(h - 1)$, where h is the Coxeter number.

More precisely, we see that $\text{Hom}_G(L(\lambda), St_n \otimes L((p^n - 1)\rho - \lambda^*)) \cong \text{Hom}_G(L(\lambda) \otimes L((p^n - 1)\rho - \lambda), St_n) \cong k$, which by virtue of the G -structure of $Q_n(\lambda)$ and the G_n -injectivity of St_n , yields that $Q_n(\lambda)$ is a (single) direct summand of $St_n \otimes L((p^n - 1)\rho - \lambda^*)$. Next we find a good filtration of this last tensor product and try to decompose it as a direct sum of indecomposables, one of which of course is $Q_n(\lambda)$.

Then again $Q_n(\lambda)$, as it is well known, is also $G(n)$ -injective, therefore the $G(n)$ -injective hull $U_n(\lambda)$ of $L(\lambda)$ is a $G(n)$ -direct summand of $Q_n(\lambda)$.

In the procedure of restriction we make extensive use of Steinberg's tensor product theorem; then we shall have to calculate lots of tensor products of simple

G -modules, which may either be done ad hoc, for example by weight considerations or by use of known results on the translation functors or, fortunately, in the case of SL_3 , by using the algorithms found in [3].

Notably for SL_3 again, in the mentioned paper [8] both the dimensions of the modules $Ext_{G(1)}^1$ and the $G(1)$ -decomposition of the $Q_1(\lambda)$'s have been explicitly calculated. Furthermore and very importantly, in [7] and much more thoroughly in [12] we find an explicit calculation of the submodule-structure of the Weyl modules (and their duals); notice that the diagrams given there are in reality virtual, since there are nowhere in them isomorphic irreducibles on the same layer. This convenience does anyway no longer hold after the mentioned restrictions.

The last Proposition above shall then be of crucial importance to articulate the way the virtual contents of the old break up and organize into the new and far more complicated diagrams, after the restrictions.

I have been working with such examples already as a young student in the late '80's; my work was then interrupted, due to outer conditions and personal obligations. After resuming my mathematical activity recently, I have found in [2] the only diagrammatic method that I know of, that somehow reminds of my own approach. There is, however, a fundamental difference from mine in theirs: Contrary to what I am doing, they start with defining abstract diagrams, and then they are looking for modules "representing" them, if any. Nonetheless that article is important, also from my point of view, especially whenever such a diagram as theirs has a unique representation ([2, 5.1]).

References

- [1] David J. Benson, Representations and Cohomology I, Cambridge Studies in Advanced Mathematics, 1991 (No. 30)
- [2] David J. Benson, Jon F. Carlson, Diagrammatic methods for modular representations and cohomology. Communications in Algebra, 01/1987; 15(1-2):53-121. DOI: 10.1080/00927878708823414
- [3] Chris D. Bowman, Stephen R. Doty, S. Martin - Decomposition of tensor products of modular irreducible representations for SL_3 : the $p \geq 5$ case. <http://arxiv.org/abs/1111.5811> (v3: 24 Sep 2014)
- [4] Jon F. Carlson, Modules and Group Algebras. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, 1996.
- [5] Charles W. Curtis, Irving Reiner, Methods of representation theory with applications to finite groups and orders, vol. 1, Wiley, 1981

- [6] E. Cline, B. Parshall, L. Scott, W. van der Kallen, Rational and generic cohomology, *Invent. Math.* 39 (1977), 143-163.
- [7] Stephen Doty and John Sullivan, The submodule structure of Weyl modules for SL_3 , *J.Alg.*, vol. 96,1, Sept. 1985, 78-93, doi:10.1016/0021-8693(85)90040-7
- [8] Henning Haahr Andersen, Extensions of simple modules for finite Chevalley groups, *J. Alg.* 111 (1987), 388–403
- [9] Peter J. Hilton, Urs Stammbach; A Course in Homological Algebra. Springer GTM (nr. 4, 1971, new edition: 1997)
- [10] Jens Carsten Jantzen, Representations of Algebraic Groups, 2. ed. Providence, RI : American Mathematical Society, 2003. 576 s.
- [11] Henning Krause, Derived categories, resolutions, and Brown representability. Lecture notes from summer school, Chicago 2004.
- [12] Kerstin Kühne-Hausmann, Zur Untermodulstruktur der Weylmoduln für SL_3 , PhD-thesis (supervisor J.C.Jantzen), Bonn 1985.
- [13] Peter Landrock, Finite Group Algebras and Their Modules. LMS Lecture Notes Series (84),Cambridge University Press (1984)